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Existence of Periodic Solutions for the Discrete-Time Counterpart of a Neutral-Type Cellular Neural Network with Time-Varying Delays and Impulses

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ABSTRACT

From the mathematical point of view, a cellular neuralnetwork (CNN) can be characterized by an array ofidentical nonlinear dynamical systems called cells (neurons) that are locally interconnected. Using thesemi-discretization method, in the present paper a discrete-time counter part of a neutral-type CNN with time-varying delays and impulses is constructed. Sufficient conditions for the existence of periodic solutions of the discrete-time system thus obtained arefoundby using thecontinuation theorem of coincidencedegreetheory.

Keywords: Cellular neutralnetworks, impulses, neutral-type CNNs, time-varying delay.

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Journal of Economic Literature (JEL) Classification:C02, C65

1. INTRODUCTION

Over the past two decades neural networks have been widely studied since they have been successfully applied to various processing problems such as optimization, image processing, associative memory and many other fields, see (Galushkin,2007; Heaton, 2011) and references given therein. Different types of applications depend on the dynamical behaviors of the neural networks. Cellular neural networks (CNNs) were introduced by Chua and Yang (1988). Since then, many researchers have done extensive and interesting works on this subject because of its potential applications in real-life problems such as signal processing, pattern recognition, chemical processes, nuclear reactors, biological systems, static image processing, associative memories, optimization problems and so on (Chua and Yang,1988;Galushkin, 2007; Haenggi, n.d.; Stamova and Ilarionov, 2010). A CNN is a massive parallel computing paradigm defined in a discrete N-dimensional space. The basic circuit unit of CNNs is called a cell (neuron). It contains linear and nonlinear circuit elements, which typically are linear capacitors, linear resistors, linear and nonlinear controlled sources, and independent sources. The structure of CNNs is similar to that found in cellular automata; namely, any cell in a CNN is connected only to its neighbor cells and the adjacent cells can interact directly with each other (Chua and Yang,1988).

Following the Chua-Yang definition, the properties and special features of the CNNs can be outlined as follows (Haenggi, n.d.):

1. A CNN is an N-dimensional regular array of entities of matrices which are called cells;
2. The cell grid can be a planar array with rectangular, triangular or hexagonal geometry, a 2D or 3D torus, a 3D finite array, or a 3D sequence of 2D arrays (layers);
3. Cells are multiple input - single output processors, all described by activation parametric functionals;
4. A cell is characterized by an internal state variable, sometimes not directly observable from outside the cell itself;
5. More than one connection network can be present, with different neighborhood sizes;
6. A CNN dynamical system can operate both in continuous-time cellular neural networks (CT-CNN) or discrete-time cellular neural networks (DT-CNN);
7. CNNs data and parameters are mostly continuous valued;
8. CNNs operate typically with more than one iteration, i.e., they are recurrent networks.

The main characteristic of CNNs is the locality of the connections between the units, that is, information is directly exchanged just between neighboring units. Of course, this characteristic also allows global processing. Communications between units not directly connected (remote) are obtained passing through other units. One of the key features of CNNs is parallel processing and global interaction of network elements.

CNN is also known as Cellular Nonlinear Network, which is an array of dynamical systems (cells) or coupled networks with local connections only.The CNNs dynamics can be described by a system of nonlinear differential equations. Using thesimplest first-order cell dynamics and linear interactions, the state equation of a cell inposition (i,j)can be represented as (Liet al., 2012; Stamova and Ilarionov, 2010):

$$\frac{dx_{ij}(t)}{dt} = -x_{ij}(t) + \sum_{(k,l)\in\mathcal{N}(i,j)} A(i,j;k,l)y_{kl}(t) + \sum_{(k,l)\in\mathcal{N}(i,j)} B(i,j;k,l)u_{kl}(t) + z(i,j;k,l),$$

where u_{ij}, x_{ij},and y_{ij} are the input, the state, and the output of the cell in position (i,j), respectively;the indices k and l denote a generic cell belonging to the neighborhood$\mathcal{N}(i,j)$ of the cell inposition (i,j). All variables are continuous. The set of matrices of the weights and the threshold $\{A,B,z\}$of the neural/nonlinear network is called the cloning template and it definesthe operation performed by the network. When the values of the cloning template do not dependon the position of the cell, the CNN is called space-invariant. In this case, the dynamic behavior ofthe network depends only on a few parameters; for instance, for a two-dimensional CNN withradius of the neighborhood$r = 1$, $A$and B are 3×3 matrices, while z is a scalar, a total of just 19 real numbers determines the CNN dynamics. The expression for the output y_{ij} is

$$y_{ij}(t) = f\left(x_{ij}(t)\right) = \frac{1}{2}\left(\left|x_{ij}(t) + 1\right| - \left|x_{ij}(t) - 1\right|\right).$$

It is possible to consider the CNN paradigm as an evolution of Cellular Automata paradigm. Moreover, it has been demonstrated that the CNN paradigm is universal, being equivalent to the Turing Machine. A mathematical description of the discrete-time case can be defined as:

$$x(t+1) = g\big(x(t)\big) + I(t) + \sum_{k=1}^{n} A\big(y_k(t), PA(j)\big) + \sum_{k=1}^{n} B\big(u_k(t), PB(j)\big), \quad j = 1,2,\dots,n, \quad y(t) = f\big(x(t)\big),$$

where$x$ is the internal state of a cell, y its output, u its external input and I a local value called bias. $A$and$B$ are two generic parametric functionals, $PA(j)$ and $PB(j)$ are the parameters arrays (typically the inter-cell connection weights).

Many researchers study the dynamics behaviors of different types of CNNs and have obtained interesting results, see (Chua and Yang,1988; Haenggi, n.d.; Stamova and Ilarionov, 2010) and references cited therein. A neural network model with delays and impulsive effects should be more accurate to describe the evolutionary process of the system. Impulsive CNNs with delays have been studied by many researchers (Heaton, 2011; Li et al., 2009; Long and Xu, 2013; Stamova and Ilarionov, 2010; Zhang and Gui, 2009) and reference given therein. In particular, Zhang and Gui (2009) employed the continuation theorem of coincidence degree theory to prove the existence of a periodic solution. Moreover, they constructed a Lyapunov function in orderto study its stability.

Neutral-type time delays appear in the study of automatic control, population dynamics and vibrating masses etc. There are not many results on neutral-type CNNs with impulses, time-varying and distributed delays.

In the present paper we construct a discrete-time counterpart of a neutral-type CNN with time-varying delays and impulses by using the semi-discretization method. We find sufficient conditions for the existence of periodic solutions of the discrete-time system thus obtained by using the aforementioned continuation theorem of coincidence degree theory.The neutral type of the system is an obstacle to the construction of a Lyapunov function to study its stability. These results were reported at the 14th International Conference of Numerical Analysis and Applied Mathematics --- ICNAAM 2016. An extended abstract (without proofs and omitting many definitions) will appear in the AIP Conference Proceedings (Akça et al., to appear).

2. A CONTINUOUS-TIME NEUTRAL-TYPE CNN AND ITS DISCRETE-TIME COUNTERPART

We consider the following impulsive neutral-type CNN with time-varying delays:

$$\dot{x}_i(t) - d_i\dot{x}_i(t-\sigma_i(t)) = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j\left(x_j(t)\right) + \sum_{j=1}^{m} c_{ij}(t)g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right) + I_i(t),$$
$$t > 0, \quad t \neq t_k, \qquad (1)$$
$$\Delta x_i(t_k) = -\alpha_{ik}x_i(t_k) + \sum_{j=1}^{m} \beta_{ijk}\Phi_j\left(x_j(t_k)\right) + \sum_{j=1}^{m} \gamma_{ijk}\Gamma_j\left(x_j\left(t_k-\tau_{ij}(t_k)\right)\right) + \zeta_{ik}, \quad k \in \mathbb{N}, \qquad (2)$$
$$x_i(s) = \varphi_i(s), \qquad s \in [-\chi, 0], \quad i = \overline{1,m}, \qquad (3)$$

where $x_i(t)$ is the state of the i-th neuron at time t and $f_j(\cdot)$, $g_j(\cdot)$denote activation functions; the functions $b_{ij}(t), c_{ij}(t)$ represent the weights (or strengths) of the synaptic connections between the j-th neuron and the i-th neuron, respectively without and with transmission delay $\tau_{ij}(t)$; $\sigma_i(t)$ is the time delay in the state velocity $\dot{x}_i(t)$; $I_i(t)$ denotes the external bias on the i-th unit at time t; $a_i(t)$ is the rate with which the i-th unit resets its potential to the equilibrium state when isolated from the network and external inputs; $t_k(k \in \mathbb{N})$are the moments (instants) of impulse effect satisfying $0 < t_1 < t_2 < \cdots < t_k < \cdots$and $\lim_{k\to\infty} t_k = \infty$; $\Delta x_i(t_k) := x_i(t_k+0) - x_i(t_k-0) \equiv x_i(t_k+0) - x_i(t_k)$represents the instantaneous change of the state of the i-th neuron at time t_k;$d_i, \alpha_{ik}, \beta_{ijk}, \gamma_{ijk}, \zeta_{ik}$aresome constants, and $\chi > 0$ will be specified later.

Now we make the following assumptions:

[**H1**] There exists a positive number ω and a positive integer p such that
$$a_i(t+\omega) = a_i(t), \quad I_i(t+\omega) = I_i(t), \quad \sigma_i(t+\omega) = \sigma_i(t) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad i = \overline{1,m},$$
$$b_{ij}(t+\omega) = b_{ij}(t), \;\; c_{ij}(t+\omega) = c_{ij}(t), \;\; \tau_{ij}(t+\omega) = \tau_{ij}(t) \;\; \text{for} \;\; t \geq 0 \;\; \text{and} \;\; i,j = \overline{1,m},$$
$$t_{k+p} = t_k + \omega \quad \text{for} \quad k \in \mathbb{N},$$
$$\alpha_{i,k+p} = \alpha_{ik}, \quad \zeta_{i,k+p} = \zeta_{ik} \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad i = \overline{1,m},$$
$$\beta_{ij,k+p} = \beta_{ijk}, \quad \gamma_{ij,k+p} = \gamma_{ijk} \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad i,j = \overline{1,m}.$$

[**H2**] $|d_i| < 1$for $i = \overline{1,m}$, $d_i = 0$ whenever $\sigma_i(t) \equiv 0$; the functions $a_i(t), b_{ij}(t), c_{ij}(t)$ are continuous on$[0,\infty]$;
$$a_i(t) > 0 \quad for \quad t \geq 0 \quad \text{and} \quad \alpha_{ik} > 0 \quad for \quad k \in \mathbb{N}, \quad \mathrm{i} = \overline{1,m}.$$

[**H3**] There exist positive constants $F_j, G_j, \mathcal{F}_j, \mathcal{G}_j (j = \overline{1,m})$ such that
$$\left|f_j(x) - f_j(y)\right| \leq F_j|x-y|, \quad \left|g_j(x) - g_j(y)\right| \leq G_j|x-y|,$$
$$\left|\Phi_j(x) - \Phi_j(y)\right| \leq \mathcal{F}_j|x-y|, \quad \left|\Gamma_j(x) - \Gamma_j(y)\right| \leq \mathcal{G}_j|x-y| \quad \text{for any } x,y \in \mathbb{R}.$$

[**H4**] The functions $\sigma_i(t)(i = \overline{1,m})$ and $\tau_{ij}(t)(i,j = \overline{1,m})$ are nonnegative, continuously differentiable for $t > 0$ and such that
$$\sup_{t>0} \dot{\sigma}_i(t) < 1, \quad \sup_{t>0} \dot{\tau}_{ij}(t) < 1 \quad for \quad i,j = \overline{1,m};$$
for each $i \in \{1,2,\ldots,m\}$, either $\sigma_i(t) \equiv 0$ or $\sigma_i(t) > 0$ for $t \in [0,\omega]$.

[**H5**] The functions $\varphi_i(s)(i = \overline{1,m})$ are piecewise continuouslydifferentiable on the interval $[-\chi, 0]$, with points of possible discontinuity of the form $t_k - \omega$, where$\chi = \max\{\sigma, \tau\}$ and
$$\sigma = \max_{i=\overline{1,m}} \sup_{t>0} \sigma_i(t), \quad \tau = \max_{i,j=\overline{1,m}} \sup_{t>0} \tau_{ij}(t).$$

The existence of periodic solutions for a system similar to (1), (3) (without impulses) under assumptions contained in **H1**-**H5** was studied in (Li et al., 2012). To find an ω-periodic solution of system (1), (2) means to determine the initial functions $\varphi_i(s)$ so that the solution of the initial value problem (1)-(3) is ω-periodic.

Similarly to our previous paper (Akça et al., 2009), henceforth we shall derive a discrete counterpart of system (1)-(3) using the semi-discretization method and obtain sufficient conditions for the existence of periodic solutions of the latter. The differentiability of the time-varying delays which is essential in the continuous-time case will not be used in this process.

For the sake of definiteness, we assume that $\chi \leq \omega$. For a positive integer N, we choose thediscretization step $h = \omega/N$. For the moment, we assume Nso large that

$$h < \min_{k=\overline{1,p}}(t_{k+1} - t_k).$$

Then each interval $[nh,(n+1)h]$ contains at most one instant of impulseeffect t_k. We also assume that

$$h < \min\left\{\inf_{t>0} \sigma_i(t) \mid \sigma_i(t) \not\equiv 0\right\}.$$

For convenience, we denote $n = [t/h]$, the greatest integer in t/h, and $n_k = [t_k/h]$. We also denote $\overline{\sigma}_i(\cdot) = [\sigma_i(\cdot)/h]$, $\overline{\tau}_{ij}(\cdot) = \left[\tau_{ij}(\cdot)/h\right]$, $N_0 = [\chi/h]$.

Let$n \in \{0\} \cup \mathbb{N}, n \neq n_k$. This means that the interval $[nh,(n+1)h]$contains no instant of impulseeffectt_k. We approximate the differential equations (1) on the interval$(nh,(n+1)h)$by

$$\dot{x}_i(t) + a_i(nh)x_i(t) - d_i[\dot{x}_i(t - \overline{\sigma}_i(nh)h) + a_i(nh)x_i(t - \overline{\sigma}_i(nh)h)] = -a_i(nh)d_i x_i\left(\left(n - \overline{\sigma}_i(nh)\right)h\right)$$
$$+I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh)f_j\left(x_j(nh)\right) + \sum_{j=1}^{m} c_{ij}(nh)g_j\left(x_j\left(\left(n - \overline{\tau}_{ij}(nh)\right)h\right)\right), \qquad i = \overline{1,m}.$$

We multiply both sides of this equation by $\exp(a_i(nh)t)$,integrate over the interval $[nh,(n+1)h]$, and then multiply by $\exp\left(-a_i(nh)(n+1)\right)$. Thus we obtain

$$x_i\left((n+1)h\right) - x_i(nh) = d_i\left[x_i\left(\left(n+1-\overline{\sigma}_i(nh)\right)h\right) - x_i\left(\left(n - \overline{\sigma}_i(nh)\right)h\right)\right] - \left(1 - e^{-a_i(nh)h}\right)x_i(nh)$$
$$+\frac{1 - e^{-a_i(nh)h}}{a_i(nh)}\left\{I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh)f_j\left(x_j(nh)\right)\right.$$
$$\left.+\sum_{j=1}^{m} c_{ij}(nh)g_j\left(x_j\left(\left(n - \overline{\tau}_{ij}(nh)\right)h\right)\right)\right\}. \qquad (4)$$

Henceforth by abuse of notation we write $x_i(n) \coloneqq x_i(nh)$ and define $\Delta x_i(n) = x_i(n+1) - x_i(n)(i = \overline{1,m}, n \in \{0\} \cup \mathbb{N})$. Forconvenience, we adopt the notations:

$$A_i(n) \coloneqq 1 - e^{-a_i(nh)h}(i = \overline{1,m},\ \ n \in \{0\} \cup \mathbb{N}\backslash\{n_k\}_{k\in\mathbb{N}}),$$
$$I_i(n) \coloneqq \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} I_i(nh)(i = \overline{1,m},\ \ n \in \{0\} \cup \mathbb{N}\backslash\{n_k\}_{k\in\mathbb{N}}),$$
$$b_{ij}(n) \coloneqq \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} b_{ij}(nh)(i,j = \overline{1,m},\ \ n \in \{0\} \cup \mathbb{N}\backslash\{n_k\}_{k\in\mathbb{N}}),$$
$$c_{ij}(n) \coloneqq \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} c_{ij}(nh)(i,j = \overline{1,m},\ \ n \in \{0\} \cup \mathbb{N}\backslash\{n_k\}_{k\in\mathbb{N}}),$$

$$\sigma_i(n) := \bar{\sigma}_i(nh)(i = \overline{1,m},\ n \in \mathbb{N}\backslash\{n_k\}_{k\in\mathbb{N}}), \quad \tau_{ij}(n) := \bar{\tau}_{ij}(nh)(i,j = \overline{1,m},\ n \in \mathbb{N}).$$

Clearly, we have $0 < A_i(n) < 1$.

Withtheabovenotation,equation (4) takes the form

$$\Delta x_i(n) = d_i\big[x_i\big(n+1-\sigma_i(n)\big) - x_i\big(n-\sigma_i(n)\big)\big] - A_i(n)x_i(n) + I_i(n) + \sum_{j=1}^{m} b_{ij}(n)f_j\left(x_j(n)\right) + \sum_{j=1}^{m} c_{ij}(n)g_j\left(x_j\left(n-\tau_{ij}(n)\right)\right), \quad i = \overline{1,m}, \quad n \neq n_k. \qquad (5)$$

Next, for $n = n_k$ the interval $[nh,(n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition (2) by

$$\Delta x_i(n_k) = -\alpha_{ik}x_i(n_k) + \zeta_{ik} + \sum_{j=1}^{m} \beta_{ijk}\Phi_j\left(x_j(n_k)\right) + \sum_{j=1}^{m} \gamma_{ijk}\Gamma_j\left(x_j\left(n_k - \tau_{ij}(n_k)\right)\right), \quad i = \overline{1,m}, \quad k \in \mathbb{N}. \qquad (6)$$

For uniformity of notation, we define

$$A_i(n_k) = \alpha_{ik}, \quad I_i(n_k) = \zeta_{ik}(i = \overline{1,m},\ k \in \mathbb{N}).$$

Now the difference system (5), (6) can be written in an operator form as

$$\Delta x = Hx, \qquad (7)$$

where

$$(Hx)_i(n) = -A_i(n)x_i(n) + I_i(n) \qquad (8)$$
$$+\begin{cases} d_i\big[x_i\big(n+1-\sigma_i(n)\big) - x_i\big(n-\sigma_i(n)\big)\big] + \sum_{j=1}^{m} b_{ij}(n)f_j\left(x_j(n)\right) + \sum_{j=1}^{m} c_{ij}(n)g_j\left(x_j\left(n-\tau_{ij}(n)\right)\right), & n \neq n_k, \\ \sum_{j=1}^{m} \beta_{ijk}\Phi_j\left(x_j(n_k)\right) + \sum_{j=1}^{m} \gamma_{ijk}\Gamma_j\left(x_j\left(n_k - \tau_{ij}(n_k)\right)\right), & n = n_k. \end{cases}$$

From the assumptions **H1**, **H2**, **H4**, it follows that

[**H6**] There exist positive integers $N$and p,such that

$$\begin{gathered} A_i(n+N) = A_i(n), \quad I_i(n+N) = I_i(n) \quad \text{for} \quad i = \overline{1,m}, \quad n \in \{0\} \cup \mathbb{N}, \\ \sigma_i(n+N) = \sigma_i(n) \quad \text{for} \quad i = \overline{1,m}, \quad n \in \{0\} \cup \mathbb{N}\backslash\{n_k\}_{k\in\mathbb{N}}, \\ \tau_{ij}(n+N) = \tau_{ij}(n) \quad \text{for} \quad i,j = \overline{1,m}, \quad n \in \{0\} \cup \mathbb{N}, \\ b_{ij}(n+N) = b_{ij}(n), \quad c_{ij}(n+N) = c_{ij}(n) \quad \text{for} \quad i,j = \overline{1,m}, \quad n \in \mathbb{N}\backslash\{n_k\}_{k\in\mathbb{N}}, \\ n_{k+p} = n_k + N \quad \text{for} \quad k \in \mathbb{N}, \\ \beta_{ij,k+p} = \beta_{ijk}, \quad \gamma_{ij,k+p} = \gamma_{ijk} \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad i,j = \overline{1,m}. \end{gathered}$$

[**H7**] $A_i(n) > 0,\ \sigma_i(n) \geq 0$ for $i = \overline{1,m},\ n \in I_N := \{0,1,\dots,N-1\},\ \tau_{ij}(n) \geq 0$for $i,j = \overline{1,m},\ n \in I_N$. Moreover, for each $i \in \{1,2,\dots,m\}$ either $\sigma_i(n) = 0$ for $n \in I_N$ or $\sigma_i(n) > 0$ for $n \in I_N$; $|d_i| < 1$ for $i = \overline{1,m}$, $d_i = 0$ whenever $\sigma_i(n) = 0$ for $n \in I_N$.

We can consider the system (7) for $n \in \{0\} \cup \mathbb{N}$, with initial conditions

$$x_i(\ell) = \varphi_i(\ell) \quad \text{for} \quad \ell = 0,-1,\dots,-N_0, \quad i = \overline{1,m}, \qquad (9)$$

where $\varphi(\ell) = \left(\varphi_1(\ell), \varphi_2(\ell), \dots, \varphi_m(\ell)\right)^T$, $\ell = 0, -1, \dots, -N_0$, are given initial vectors $\left(\ell = \left[\frac{s}{h}\right], \varphi_i(\ell) := \varphi_i(\ell h)\right)$. To find an N-periodic solution of system (7) means to determine the initial vectors $\varphi(\ell)$ so that the solution of the initial-value problem (7), (9) is N-periodic.

3. MAIN RESULT

In order to formulate our main result, we introduce some notation:
:
For an$N$-periodic sequence $v(n)$,we denote $\tilde{v} = \sum_{n=0}^{N-1} v(n)$;

$$\bar{b}_{ij} = \sup_{n \neq n_k} |b_{ij}(n)|, \quad \bar{c}_{ij} = \sup_{n \neq n_k} |c_{ij}(n)|, \quad \bar{\beta}_{ij} = \max_{\overline{k=1,p}} |\beta_{ijk}|, \quad \bar{\gamma}_{ij} = \max_{\overline{k=1,p}} |\gamma_{ijk}|, \quad i,j = \overline{1,m};$$

$$\rho_i = |\tilde{I}_i| + (N-p) \sum_{j=1}^{m} \left(\bar{b}_{ij} |f_j(0)| + \bar{c}_{ij} |g_j(0)|\right) + p \sum_{j=1}^{m} \left(\bar{\beta}_{ij} |\Phi_j(0)| + \bar{\gamma}_{ij} |\Gamma_j(0)|\right), \quad i = \overline{1,m}; \tag{10}$$

$$\mathcal{B}_{ij} = (N-p)\left(\bar{b}_{ij} F_j + \bar{c}_{ij} G_j\right) + p\left(\bar{\beta}_{ij} \mathcal{F}_j + \bar{\gamma}_{ij} \mathcal{G}_j\right), \quad i,j = \overline{1,m}. \tag{11}$$

Next we introduce the conditions

[**H8**] For $i = \overline{1,m}$, we have

$$\tilde{A}_i \frac{1 - \tilde{A}_i}{1 + \tilde{A}_i} > 2(N-p)|d_i|.$$

[**H9**] $\min_{\overline{i=1,m}} \left(\tilde{A}_i - \sum_{j=1}^{m} \mathcal{B}_{ji}\right) > 0.$

We introduce the $m \times m$ matrices

$$\mathcal{A} = \operatorname{diag}\left(\tilde{A}_i \frac{1 - \tilde{A}_i}{1 + \tilde{A}_i} - 2(N-p)|d_i|, \quad i = \overline{1,m}\right), \quad \mathcal{B} = \left(\mathcal{B}_{ij}\right)_{i,j=1}^{m}, \tag{12}$$

and the condition

[**H10**] The matrix $\mathcal{A} - \mathcal{B}$ is an *M*-matrix.

This condition implies that the matrix $\mathcal{A} - \mathcal{B}$ is nonsingular and its inverse has only nonnegative entries (Berman and Plemmons, 1979; Fiedler, 1986).

Now, we can state our main result as the following theorem:

Theorem 1. *Suppose that conditions* **H3**, **H6**-**H10** *hold. Then the equation* (7) *has at least one* N*-periodic solution.*

Proof. We shall prove this theorem using Mawhin's continuation theorem (Gaines and Mawhin, 1977, p. 40). To state this theorem, we need some preliminaries, presented as in (Akça et al., 2009; Li et al,, 2012):

Let $\mathbb{X}, \mathbb{Y}$ be real Banach spaces, $L: \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ be a linear mapping, and $H: \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$and $\operatorname{Im} L$ is closed in $\mathbb{Y}$. If $L$is a Fredholm mapping of index zero and there exist continuous projectors $P: \mathbb{X} \to \mathbb{X}$ and $Q: \mathbb{Y} \to \mathbb{Y}$such that$\operatorname{Im} P = \operatorname{Ker} L$,$\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, then the mapping$L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}: (I-P)\mathbb{X} \to \operatorname{Im} L$ is invertible. We denote the inverse of this mapping byK_P. If $\Omega$is an openbounded subset of$\mathbb{X}$, the mapping Hwill be called L-compacton $\bar{\Omega}$ if $QH(\bar{\Omega})$ is bounded and

$K_P(I-Q)H: \overline{\Omega} \to \mathbb{X}$is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J: \operatorname{Im} Q \to \operatorname{Ker} L$.

Now,Mawhin's continuation theorem can be stated as follows:

Lemma 1.*Let* L*be a Fredholm mapping of index zero, let*$\Omega \subset \mathbb{X}$*be an open bounded set and let*$H: \mathbb{X} \to \mathbb{Y}$*be a continuous operator which is*L*-compact on*$\overline{\Omega}$*. Assume that the following conditions hold*:

(*a*) *for each*$\lambda \in (0,1), x \in \partial\Omega \cap \operatorname{Dom} L,\ Lx \neq \lambda Hx$;

(*b*) *for each*$x \in \partial\Omega \cap \operatorname{Ker} L,\ QHx \neq 0$;

(*c*)$\deg(JQH, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where*$\deg(\cdot,\cdot,\cdot)$*is the Brouwer degree*.

Then the equation$Lx = Hx$*has at least one solutionin*$\overline{\Omega} \cap \operatorname{Dom} L$.

It is much easier to apply this lemma to difference equations than to differential equations since in the former case all spaces are finite dimensional.

Before we proceed further, we shall recall the definition of Brouwer degree (Milnor, 1969).

Suppose that M and N are two oriented differentiable manifolds of dimension n (without boundary) with M compact and N connected and suppose that $f: M \to N$ is a differentiable mapping. Let $Df(x)$ denote the differential mapping at the point$x \in M$, that is, the linear mapping $Df(x): T_x(M) \to T_{f(x)}(N)$. Let $\operatorname{sign} Df(x)$ denote the sign of the determinant of $Df(x)$. That is, the sign is positive if f preserves orientation and negative if f reverses orientation.

Definition 1. Let $y \in N$be a regular value, then we define the Brouwerdegree (or just degree) of f by

$$\deg f \equiv \deg(f, M, y) := \sum_{x \in f^{-1}(y)} \operatorname{sign} Df(x).$$

It can be shown that the degree does not depend on the regular value $y$that we pick so that $\deg f$ is well defined.Note that this degree coincides with the degree as defined for maps of spheres.

Let us choose $\mathbb{X} = \mathbb{Y} = \left\{x(n) = \left(x_1(n), x_2(n), \dots, x_m(n)\right)^T : x(n+N) = x(n),\ n \in \{0\} \cup \mathbb{N}\right\}$. If we define$|x_i| = \max_{n \in I_N} |x_i(n)|,\ \|x\| = \sum_{i=1}^{m} |x_i|$, then $\mathbb{X}$ is a Banach space with the norm $\|\cdot\|$. For$x \in \mathbb{X}$, let Hx be defined by (8), $Lx = \Delta x$and

$$Px = Qx = \frac{1}{N}\tilde{x} = \frac{1}{N}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m)^T.$$

Then, $\operatorname{Ker} L = \{x \in \mathbb{X}: x = h \in \mathbb{R}^m\}$(vectors with componentsindependent of$n$), $\operatorname{Im} L = \{x \in \mathbb{X}: \sum_{n=0}^{N-1} x_i(n) = 0,\ i = \overline{1,m}\}$is a closed set in $\mathbb{X}$, and$\operatorname{codim} L = m$. Thus,$L$ is a Fredholm mapping of index zero. It is easy to see that $P$and $Q$are continuous projectors and $\operatorname{Im} P = \operatorname{Ker} L, \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$, and H is L-compact on $\overline{\Omega}$ for any bounded set $\Omega \subset \mathbb{X}$. Moreover, in condition (*c*) of Lemma 1 the isomorphism Jcan be taken as the identity operator$I$.

Now, we will derive some estimates for the solutions x of the operator equation $Lx = \lambda Hx$for $\lambda \in (0,1)$, that is,

$$\Delta x_i(n) = \lambda(Hx)_i(n), \quad n \in I_N, \quad i = \overline{1,m}, \tag{13}$$

First, from (13) and (8) for $n \neq n_k$ we obtain

$$|\Delta x_i(n)| \leq |d_i|\big(|x_i(n+1-\sigma_i(n))| + |x_i(n-\sigma_i(n))|\big) + A_i(n)|x_i(n)|$$
$$+|I_i(n)| + \left|\sum_{j=1}^{m} b_{ij}(n) f_j\left(x_j(n)\right)\right| + \left|\sum_{j=1}^{m} c_{ij}(n) g_j\left(x_j\left(n-\tau_{ij}(n)\right)\right)\right|$$
$$\leq \big(2|d_i| + A_i(n)\big)|x_i| + |I_i(n)| + \sum_{j=1}^{m} \bar{b}_{ij}\big(F_j|x_j(n)| + |f_j(0)|\big) + \sum_{j=1}^{m} \bar{c}_{ij}\left(G_j\left|x_j\left(n-\tau_{ij}(n)\right)\right| + |g_j(0)|\right)$$
$$\leq \big(2|d_i| + A_i(n)\big)|x_i| + |I_i(n)| + \sum_{j=1}^{m}\big(\bar{b}_{ij}|f_j(0)| + \bar{c}_{ij}|g_j(0)|\big) + \sum_{j=1}^{m}\big(\bar{b}_{ij}F_j + \bar{c}_{ij}G_j\big)|x_j|.$$

Similarly, for $n = n_k$ we have

$$|\Delta x_i(n_k)| \leq A_i(n_k)|x_i(n_k)| + |I_i(n_k)| + \left|\sum_{j=1}^{m} \beta_{ijk}\Phi_j\left(x_j(n_k)\right)\right| + \left|\sum_{j=1}^{m} \gamma_{ijk}\Gamma_j\left(x_j\left(n_k-\tau_{ij}(n_k)\right)\right)\right|$$
$$\leq A_i(n_k)|x_i| + |I_i(n_k)| + \sum_{j=1}^{m} \bar{\beta}_{ij}\big(\mathcal{F}_j|x_j(n_k)| + |\Phi_j(0)|\big) + \sum_{j=1}^{m} \bar{\gamma}_{ij}\left(\mathcal{G}_j\left|x_j\left(n_k-\tau_{ij}(n_k)\right)\right| + |\Gamma_j(0)|\right)$$
$$\leq A_i(n_k)|x_i| + |I_i(n_k)| + \sum_{j=1}^{m}\big(\bar{\beta}_{ij}|\Phi_j(0)| + \bar{\gamma}_{ij}|\Gamma_j(0)|\big) + \sum_{j=1}^{m}\big(\bar{\beta}_{ij}\mathcal{F}_j + \bar{\gamma}_{ij}\mathcal{G}_j\big)|x_j|.$$

From the above inequalities, we obtain

$$\sum_{n=0}^{N-1}|\Delta x_i(n)| \leq \big(2(N-p)|d_i| + \tilde{A}_i\big)|x_i| + |\widetilde{I_i}| + (N-p)\sum_{j=1}^{m}\big(\bar{b}_{ij}|f_j(0)| + \bar{c}_{ij}|g_j(0)|\big)$$
$$+p\sum_{j=1}^{m}\big(\bar{\beta}_{ij}|\Phi_j(0)| + \bar{\gamma}_{ij}|\Gamma_j(0)|\big) + \sum_{j=1}^{m}\big[(N-p)\big(\bar{b}_{ij}F_j + \bar{c}_{ij}G_j\big) + p\big(\bar{\beta}_{ij}\mathcal{F}_j + \bar{\gamma}_{ij}\mathcal{G}_j\big)\big]|x_j|$$

or, using the notations (10) and (11),

$$\sum_{n=0}^{N-1}|\Delta x_i(n)| \leq \big(2(N-p)|d_i| + \tilde{A}_i\big)|x_i| + \rho_i + \sum_{j=1}^{m}\mathcal{B}_{ij}|x_j|. \tag{14}$$

Adding together all equations of (13) for $n \in I_N$, we obtain

$$\sum_{n=0}^{N-1} A_i(n)x_i(n) = d_i \sum{}' \left(x_i(n+1-\sigma_i(n)) - x_i(n-\sigma_i(n))\right) + \sum_{n=0}^{N-1} I_i(n)$$
$$+\sum_{j=1}^{m}\Bigg\{\sum{}' \left[b_{ij}(n) f_j\left(x_j(n)\right) + c_{ij}(n) g_j\left(x_j\left(n-\tau_{ij}(n)\right)\right)\right]$$
$$+\sum_{k=1}^{p}\left[\beta_{ijk}\Phi_j\left(x_j(n_k)\right) + \gamma_{ijk}\Gamma_j\left(x_j\left(n_k - \tau_{ij}(n_k)\right)\right)\right]\Bigg\},$$

where by definition

$$\sum{}' v(n) = \sum_{n=0}^{N-1} v(n) - \sum_{k=1}^{p} v(n_k)$$
$$= v(0) + \cdots + v(n_1 - 1) + v(n_1+1) + \cdots + v(n_p - 1) + v(n_p+1) + \cdots + v(N-1),$$

that is, Σ' is a sum over the set $I_N \backslash \{n_k\}_{k=1}^{p}$.

As above, we obtain

$$\left|\sum_{n=0}^{N-1} A_i(n)x_i(n)\right| \leq 2(N-p)|d_i||x_i| + \rho_i + \sum_{j=1}^{m} \mathcal{B}_{ij}|x_j|. \quad (15)$$

Now, we shall use the following lemma (Fan and Wang, 2002; Xu et al., 2005):

Lemma 2. *Let $v: \mathbb{Z} \to \mathbb{Z}$ be N-periodic, i.e., $v(n+N) = v(n)$ for any $n \in \mathbb{Z}$. Then, for any fixed $\nu_1, \nu_2 \in I_N$ and any $n \in \mathbb{Z}$ we have*

$$v(\nu_2) - \sum_{k=0}^{N-1} |v(k+1) - v(k)| \leq v(n) \leq v(\nu_1) + \sum_{k=0}^{N-1} |v(k+1) - v(k)|.$$

According to Lemma 2, for arbitrary $n, \nu_1, \nu_2 \in I_N$ we have

$$x_i(\nu_2) - \sum_{n=0}^{N-1} |\Delta x_i(n)| \leq x_i(n) \leq x_i(\nu_1) + \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

We multiply these inequalities by $A_i(n)$ and sum up over I_N to obtain

$$\tilde{A}_i x_i(\nu_2) - \tilde{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)| \leq \sum_{n=0}^{N-1} A_i(n)x_i(n) \leq \tilde{A}_i x_i(\nu_1) + \tilde{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

Let $|x_i(\nu_0)| = |x_i| = \max_{n \in I_N} |x_i(n)|$. If $x_i(\nu_0) \geq 0$, we choose $\nu_2 = \nu_0$. Then,

$$\tilde{A}_i|x_i| = \tilde{A}_i x_i(\nu_2) \leq \sum_{n=0}^{N-1} A_i(n)x_i(n) + \tilde{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

If $x_i(\nu_0) < 0$, we choose $\nu_1 = \nu_0$,

$$\tilde{A}_i|x_i| = -\tilde{A}_i x_i(\nu_1) \leq -\sum_{n=0}^{N-1} A_i(n)x_i(n) + \tilde{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

Thus, in both cases we have

$$\tilde{A}_i|x_i| \leq \left|\sum_{n=0}^{N-1} A_i(n)x_i(n)\right| + \tilde{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

Making use of the estimates (14) and (15), we obtain

$$\begin{aligned}\tilde{A}_i|x_i| &\leq 2(N-p)|d_i||x_i| + \rho_i + \sum_{j=1}^{m} \mathcal{B}_{ij}|x_j| + \tilde{A}_i \left\{ \left(2(N-p)|d_i| + \tilde{A}_i\right)|x_i| + \rho_i + \sum_{j=1}^{m} \mathcal{B}_{ij}|x_j| \right\} \\ &= \left\{2\left(1+\tilde{A}_i\right)(N-p)|d_i| + \tilde{A}_i^2\right\}|x_i| + \left(1+\tilde{A}_i\right)\left(\rho_i + \sum_{j=1}^{m} \mathcal{B}_{ij}|x_j|\right)\end{aligned}$$

or

$$\left(\tilde{A}_i \frac{1-\tilde{A}_i}{1+\tilde{A}_i} - 2(N-p)|d_i|\right)|x_i| - \sum_{j=1}^{m} \mathcal{B}_{ij}|x_j| \leq \rho_i. \quad (16)$$

If we introduce the vectors $|\boldsymbol{x}| = (|x_1|, |x_2|, \dots, |x_m|)^T$ and $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_m)^T$, then the system of inequalities (16) for $i = \overline{1, m}$ can be written in a matrix form

$$(\mathcal{A} - \mathcal{B})|\boldsymbol{x}| \leq \boldsymbol{\rho}, \quad (17)$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ were introduced in (12). By virtue of condition **H10**, the inequality (17) implies

$$|\boldsymbol{x}| \leq (\mathcal{A}-\mathcal{B})^{-1}\boldsymbol{\rho}.$$

If $(\mathcal{A}-\mathcal{B})^{-1}\boldsymbol{\rho} = (C_1^*, C_2^*, \dots, C_m^*)^T$, this means that the components of each solution of $\Delta x = \lambda Hx$ satisfy $|x_i| \leq C_i^*$. If we denote $C^* = \sum_{i=1}^m C_i^*$, then each solution of $\Delta x = \lambda Hx$ satisfies $\|x\| \leq C^*$.

Now, we take $\Omega = \{x \in \mathbb{X}: \|x\| < C\}$, where $C > C^*$ will be chosen later. Obviously, Ω satisfies condition (*a*) of Lemma 1.

Next, let $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^m$, i.e., x is a constant vector in $\mathbb{R}^m$ with $\|x\| = C$. For such x,

$$(Hx)_i(n) = -A_i(n)x_i + I_i(n) + \sum_{j=1}^m b_{ij}(n)f_j(x_j) + \sum_{j=1}^m c_{ij}(n)g_j(x_j), \quad n \neq n_k,$$
$$(Hx)_i(n_k) = -A_i(n_k)x_i + I_i(n_k) + \sum_{j=1}^m \beta_{ijk}\Phi_j(x_j) + \sum_{j=1}^m \gamma_{ijk}\Gamma_j(x_j).$$

Then,

$$N(QHx)_i = -\tilde{A}_i x_i + \tilde{I}_i + \sum_{j=1}^m \left\{ \sum\nolimits' \left[b_{ij}(n)f_j(x_j) + c_{ij}(n)g_j(x_j)\right] + \sum_{k=1}^p \left[\beta_{ijk}\Phi_j(x_j) + \gamma_{ijk}\Gamma_j(x_j)\right] \right\}$$

and

$$N|(QHx)_i| \geq \tilde{A}_i|x_i| - |\tilde{I}_i| - \sum_{j=1}^m \left\{ \sum\nolimits' \left[|b_{ij}(n)|F_j + |c_{ij}(n)|G_j\right] + \sum_{k=1}^p \left[|\beta_{ijk}|\mathcal{F}_j + |\gamma_{ijk}|\mathcal{G}_j\right] \right\} |x_j|$$
$$- \sum_{j=1}^m \left\{ \sum\nolimits' \left[|b_{ij}(n)| \cdot |f_j(0)| + |c_{ij}(n)| \cdot |g_j(0)|\right] + \sum_{k=1}^p \left[|\beta_{ijk}| \cdot |\Phi_j(0)| + |\gamma_{ijk}| \cdot |\Gamma_j(0)|\right] \right\}$$
$$\geq \tilde{A}_i|x_i| - |\widetilde{I_i}| - \sum_{j=1}^m \left\{ (N-p)\left[\bar{b}_{ij}F_j + \bar{c}_{ij}G_j\right] + p\left[\bar{\beta}_{ij}\mathcal{F}_j + \bar{\gamma}_{ij}\mathcal{G}_j\right] \right\} |x_j|$$
$$- \sum_{j=1}^m \left\{ (N-p)\left[\bar{b}_{ij}|f_j(0)| + \bar{c}_{ij}|g_j(0)|\right] + p\left[\bar{\beta}_{ij}|\Phi_j(0)| + \bar{\gamma}_{ij}|\Gamma_j(0)|\right] \right\}$$
$$= \tilde{A}_i|x_i| - \sum_{j=1}^m \mathcal{B}_{ij}|x_j| - \rho_i.$$

Thus,

$$N\|QHx\| = \sum_{i=1}^m N|(QHx)_i| \geq \sum_{i=1}^m \tilde{A}_i|x_i| - \sum_{i=1}^m \sum_{j=1}^m \mathcal{B}_{ij}|x_j| - \sum_{i=1}^m \rho_i$$
$$= \sum_{i=1}^m \left(\tilde{A}_i - \sum_{j=1}^m \mathcal{B}_{ji} \right) |x_i| - \sum_{i=1}^m \rho_i \geq \min_{i=\overline{1,m}} \left(\tilde{A}_i - \sum_{j=1}^m \mathcal{B}_{ji} \right) \|x\| - \sum_{i=1}^m \rho_i = \min_{i=\overline{1,m}} \left(\tilde{A}_i - \sum_{j=1}^m \mathcal{B}_{ji} \right) C - \sum_{i=1}^m \rho_i.$$

By condition **H9**,

$$\min_{i=\overline{1,m}} \left(\tilde{A}_i - \sum_{j=1}^m \mathcal{B}_{ji} \right) > 0.$$

Then we can choose $C > C^*$ so large that

$$\min_{i=\overline{1,m}} \left(\tilde{A}_i - \sum_{j=1}^m \mathcal{B}_{ji} \right) C > \sum_{i=1}^m \rho_i.$$

Hence, for $x \in \partial\Omega \cap \operatorname{Ker} L$ we have $N\|QHx\| > 0$ and $QHx \neq 0$, that is, condition (*b*) of Lemma 1 is satisfied.

To prove (*c*), we define the mapping $(QH)_\mu: \operatorname{Dom} L \times [0.1] \to \mathbb{X}$ by $(QH)_\mu = -\frac{\mu}{N}\tilde{A} + (1-\mu)QH$, where $\tilde{A}x = \left(\tilde{A}_1 x_1, \tilde{A}_2 x_2, \dots, \tilde{A}_m x_m\right)^T$.

For $x \in \partial\Omega \cap \operatorname{Ker} L$, we have

$$\big((QH)_{\mu}x\big)_i = -\frac{1}{N}\tilde{A}_i x_i + \frac{1-\mu}{N}\left\{\tilde{I}_i + \sum_{j=1}^{m}\left[\sum{}' \,\Big(b_{ij}(n)f_j(x_j) + c_{ij}(n)g_j(x_j)\Big) + \sum_{k=1}^{p}\Big(\beta_{ijk}\Phi_j(x_j) + \gamma_{ijk}\Gamma_j(x_j)\Big)\right]\right\}.$$

As above, we obtain

$$\|(QH)_{\mu}x\| \geq \frac{1}{N}\left\{\min_{i=1,m}\left(\tilde{A}_i - \sum_{j=1}^{m}\mathcal{B}_{ji}\right)C - \sum_{i=1}^{m}\rho_i\right\} > 0.$$

This means that $(QH)_{\mu}x \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$and $\mu \in [0,1]$. From the homotopy invariance of the Brouwer degree, it follows that

$$\deg(QH, \Omega \cap \operatorname{Ker} L, 0) = \deg\left(-\frac{1}{N}\tilde{A}, \Omega \cap \operatorname{Ker} L, 0\right) = (-1)^m \neq 0.$$

According to Lemma 1, equation (7) has at least one N-periodic solution. This completes the proof of Theorem 1. □

4. CONCLUSION

In the present paper, we first provided a short overview on cellular neural networks. Next, by using the semi-discretization method, we constructed its discrete-time counterpart. Then, we found sufficient conditions for the existence of periodic solutions of the discrete-time system thus obtained by using the continuation theorem of coincidence degree theory.

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